Periodic orbits and topological entropy of delayed maps

E. Ferretti Manffra* and H. Kantz
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, 01187 Dresden, Germany

W. Just†
Queen Mary and Westfield College, Mile End Road, London E1 4NS, United Kingdom
(Received 25 July 2000; published 20 March 2001)

The periodic orbits of a nonlinear dynamical system provide valuable insight into the topological and metric properties of its chaotic attractors. In this paper we describe general properties of periodic orbits of dynamical systems with feedback delay. In the case of delayed maps, these properties enable us to provide general arguments about the boundedness of the topological entropy in the high delay limit. As a consequence, all the metric entropies can be shown to be bounded in this limit. The general considerations are illustrated in the cases of Bernoulli-like and Hénon-like delayed maps.

DOI: 10.1103/PhysRevE.63.046203 PACS number(s): 05.45.Jn, 02.30.Ks

I. INTRODUCTION

As the mechanism of retarded feedback is present in almost all situations associated with finite propagation times, delayed differential equations of the form

$$\dot{\vec{x}}=\vec{F}(\vec{x}(t),\vec{x}(t-\tau)) \tag{1}$$

are largely employed to model control processes in the nature and technology. Examples of successful models using delayed differential equations can be found in the fields of physiology [1,2], biology [3], laser physics [4], and economy [5]. Recently, a better treatment of experimental data from delayed systems has been made possible by methods that allow a complete and accurate characterization of the chaotic attractors [6]. Besides this relevance in the applied sciences, delayed dynamical systems have interesting dynamical properties. The phase space of Eq. (1) is infinite dimensional and therefore high dimensional chaotic attractors may appear. Additionally it is found empirically that the Lyapunov dimension grows linearly with the delay while the metric entropy remains bounded [7–9] in the large delay limit. This behavior is related to the scaling properties of the Lyapunov spectrum.

In Ref. [10], a conjecture is proposed to explain the scaling behavior of the Kaplan-Yorke dimension for delayed systems of the form

$$\dot{x}(t)=\gamma x(t)+f(x(t-\tau)). \tag{2}$$

The authors observe that the autocorrelation time of the feedback (denoted as δ) is independent and much smaller than τ in the limit $\gamma\tau\gg1$, with $\gamma<1$. They conjecture that the number of active degrees of freedom would then be proportional to τ/δ which explains the linear increasing of the dimension with the delay value. The entropy is considered as the average amount of information stored during the memory time δ and this explains also the independence of the entropy with respect to the delay time.

More recently, delayed systems with nonlinear instantaneous coupling are investigated [11] [systems for which the first term on the right-hand side (RHS) of Eq. (2) is also nonlinear]. These systems may exhibit anomalous Lyapunov exponents, which do not depend on the delay value, in contrast to Eq. (2) (whose exponents scale like the inverse of the delay value). For Bernoulli-like delayed maps, the authors of [11] derive a closed form of the Lyapunov exponents spectrum in the limit of high delays. By summing up the positive part of this spectrum the boundedness of the metric entropy can be shown. The same argument is valid for any piecewise linear map with constant Jacobian.

In the present work we provide more general arguments for the boundedness of the metric entropy of chaotic attractors in delayed systems. We restrict our analysis to delayed maps and construct the arguments on the basis of periodic orbits. It is well known that the unstable periodic orbits form a skeleton of the chaotic motion and that the natural measure can be characterized by these orbits under certain conditions [12,13]. Therefore it is natural to expect that the study of periodic orbits of delayed systems provides insight in their topological and metric properties. Indeed, through general properties of periodic orbits of delayed systems, we provide an heuristic argument on the boundedness of topological entropy in the limit of large delays. In the case of piecewise linear delayed maps a more rigorous argument is provided.

The paper is organized as follows. In Sec. II we describe the properties of periodic orbits for a general delayed map and basing on them propose a bound for the topological entropy. In Sec. III, piecewise linear delayed maps and their topological entropies are discussed using the Bernoulli-like map as an example. In Sec. IV the topological entropy is estimated for an Hénon-like delayed map and in Sec. V we present discussions and conclusions.

II. PERIODIC ORBITS AND THE TOPOLOGICAL ENTROPY

Consider a general form of a delayed map with a single delay,

*Electronic address: elis@mpipks-dresden.mpg.de
†Electronic address: W.Just@qmw.ac.uk

$$x_{n+1}=F(x_n,x_{n-T}), \tag{3}$$

where T is an integer delay value. In contrast to finite dimensional flows, the technique of the Poincaré surface of section does not yield, in general, a direct relationship between delayed maps and delayed differential equations. In spite of this, one might argue that delayed maps share essential ergodic properties with delayed time continuous equations of motion.

A p-periodic orbit of the system (3) is given by a sequence of points $\{\overline{x}_0,\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_{p-1}\}$ (later called periodic points), which repeats in time as Eq. (3) is iterated. These points obey the equations

$$\overline{x}_{i+1}=F(\overline{x}_i,\overline{x}_{i-T}) \tag{4}$$

for $0\leqslant i<p-1$, with the boundary condition

$$\overline{x}_0=F(\overline{x}_{p-1},\overline{x}_{p-1-T}), \tag{5}$$

where the indices are understood modulo the period p.

For any p, Eqs. (4) and (5) are invariant under the transformation $T\mapsto T+np$, $\quad n\in\mathbf{Z}$ of the delay value. Period-p orbits found for a delay value T will be exactly the same as those for $\tilde{T}=T+np$, as long as $\tilde{T}$ and T are both positive. As a consequence, one has the following relation for the p-periodic points of the map:

$$N(p,T)=N(p,T+np), \tag{6}$$

where $N(p,T)$ denotes the number of p-periodic points for a delay T. It is possible to show that a similar property is also valid for delayed differential equations [14].

We are going to use this relation to estimate the topological entropy of the map (3). It is well known that under certain mathematical conditions the topological entropy can be related to the number of periodic points of a map [15,16]:

$$h(T)=\limsup_{p\to\infty}\frac{\ln N(p,T)}{p}. \tag{7}$$

As a consequence of Eq. (6), we derive a heuristic argument, why the topological entropy should be bounded in the limit of large delay. We simply insert the asymptotic relation $N(p,T)\simeq\exp[ph(T)]$ in Eq. (6),

$$\exp[ph(T)]\simeq\exp[ph(T+np)]=\exp[ph(\tilde{T})]. \tag{8}$$

In the limit $p\to\infty$, also $\tilde{T}\to\infty$ so that the topological entropy must be bounded. But this argument is very rough since it suggests that the entropy does not depend on T at all.

To improve this consideration we have to take into account prefactors. For a finite period p, we have the relation

$$N(p,T)=C(p,T)\exp[ph(T)], \tag{9}$$

where in view of Eq. (7) the prefactor obeys the constraint

$$\limsup_{p\to\infty}\frac{\ln C(p,T)}{p}=0, \tag{10}$$

i.e., it depends on p weaker than exponentially. If we combine Eq. (9) and Eq. (6) we get the exact equations

$$\begin{aligned}C(p,T)\exp[ph(T)]&=N(p,T)\\&=N(p,T+2p)\\&=C(p,T+2p)\exp[ph(T+2p)]\end{aligned} \tag{11}$$

and

$$\begin{aligned}C(2p,T)\exp[2ph(T)]&=N(2p,T)\\&=N(2p,T+2p)\\&=C(2p,T+2p)\exp[2ph(T+2p)].\end{aligned} \tag{12}$$

Hence

$$\frac{C(2p,T)}{C(p,T)}\exp[ph(T)]=\frac{C(2p,T+2p)}{C(p,T+2p)}\exp[ph(T+2p)] \tag{13}$$

follows. Therefore our final result can be written as

$$\begin{aligned}h(T+2p)=h(T)&+\frac{\ln C(2p,T)-\ln C(p,T)}{p}\\&-\frac{\ln C(2p,T+2p)-\ln C(p,T+2p)}{p}.\end{aligned} \tag{14}$$

If the third term in the RHS of Eq. (14) is bounded, one should expect that in the limit $p\to\infty$ the entropy $h(\infty)$ is finite.

All our previous expressions are valid for arbitrary T and p. In our final argument we have used the assumption that to some extent the limit (10) is uniform in the delay time. Such an assumption cannot be proven in the general case and we will have a closer look on this issue within the discussion of our examples. Nevertheless, under such a quite general assumption the topological entropy remains bounded in the limit of large delay time.

III. PIECEWISE LINEAR MAPS—THE BERNOULLI SHIFT

In order to illustrate the ideas above, we show the results concerning periodic orbits of a specific delayed map: the Bernoulli shift studied in Ref. [11]. For this purpose we restrict to a special type of delayed map, namely,

$$x_{n+1}=(1-\epsilon)F(x_n)+\epsilon F(x_{n-T}), \tag{15}$$

which mimics to some extent the coupling known from unidirectional coupled map lattices. Here the parameter ϵ governs the strength of the delay term. The special structure of Eq. (15) ensures that the dynamics is well defined irrespective of the type of the particular map.

There are two simple cases where the topological entropy can be evaluated by inspection. First considering $\epsilon=0$, the system turns out to be a one-dimensional map and the topological entropy is equal to that of the map F. In the opposite case, $\epsilon=1$, Eq. (15) reduces to $T+1$ independent copies of the map F acting on the time scale $T+1$. Hence, if $N(k,0)$ denotes the number of period-k points of the map F, then by combinatorics (15) with $\epsilon=1$ has $N(p,T)=N(k,0)^{T+1}$ periodic points of period $p=k(T+1)$. Taking the limit $k\to\infty$, Eq. (7) yields again the topological entropy, per unit time, of the single map F.

For intermediate values of ϵ, no general reasoning seems to be available. We, therefore, now specialize to the Bernoulli map

$$F(x)=2x-\mathrm{sgn}(x), \quad x\in[-1,1]. \tag{16}$$

Since the map is piecewise linear the periodic points of Eq. (15) can be easily estimated. Consider an orbit of period $p,\overline{x}_0,\overline{x}_1,\ldots,\overline{x}_{p-1}$. Then by $\sigma_k:=\mathrm{sgn}(\overline{x}_k)$ we can assign a period-p symbol sequence to this orbit. This assignment is injective, i.e., there exists at most one period-p orbit for each period-p symbol sequence: combining Eq. (15) and Eq. (16) the periodic orbit is determined by

$$\overline{x}_{n+1}=2(1-\epsilon)\overline{x}_n+\epsilon\overline{x}_{n-T}-(1-\epsilon)\sigma_n-\epsilon\sigma_{n-T} \tag{17}$$

and such a linear inhomogeneous equation has at most one solution that satisfies the self-consistency condition $\mathrm{sgn}(\overline{x}_k)=\sigma_k$ for a given symbol sequence $\sigma_0,\sigma_1,\ldots,\sigma_{p-1}$. Hence the number of period-p points obeys

$$N(p,T)\leqslant 2^p \tag{18}$$

and the topological entropy of the single map F yields an upper bound for the entropy of Eq. (15).

Since the topological entropy yields an upper bound for any type of Kolmogorov Sinai entropy [15], the result implies that all these entropies are bounded in the limit of large delay. Such a result is in accordance with results obtained on the basis of Lyapunov spectra [11] and illustrates that the entropies in contrast with dimensions do not increase with the delay time.

In Fig. 1 we compare numerically obtained values of the topological entropy and the Kolmogorov Sinai entropy for $T=1$ and different values of ϵ.

For this particular model the exact values of both entropies coincide since the Jacobian is constant. A proof for this statement is based on the absence of multifractality in the system, i.e., all the Rényi entropies can be shown to have the same value. Thus the difference visible in Fig. 1 yields the accuracy for the method by which the topological entropy was estimated. Apart from deviations near the minimum, the topological entropy was accurately recovered. In fact near such a minimum we expect quite dramatic topological changes that prevent a good convergence of the estimates of h_{top} based on Eq. (7).

A slightly more detailed analysis is possible based on a numerical evaluation of Eq. (17). Here a severe kind of pruning can be detected which is related to the time scale set by the delay T. First of all, if we consider periodic points of period $p=T$, then because of $\overline{x}_{n-T}=\overline{x}_n$ Eq. (15) reduces to the single map F. Hence the delayed system admits the same number of periodic points of order $p=T$ as the single map F. A similar feature occurs for periods $p=T+1$. Here because of $\overline{x}_{n-T}=\overline{x}_{n+1}$, Eq. (17) reduces to

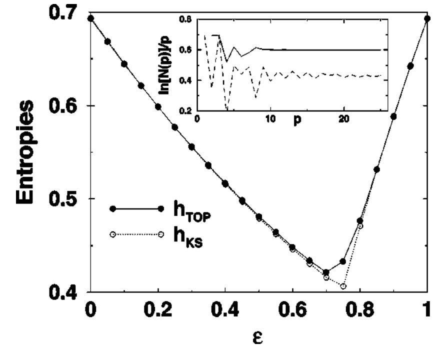

FIG. 1. Estimated topological and metric entropies (h_{top} and h_{ks}, respectively) for the Bernoulli shift with $T=1$. Metric entropies have been estimated from the Lyapunov spectrum using Pesins identity and topological entropy by $\ln N(p,1)/p$ with $p=25$. The inset shows the estimates of topological entropy as a function of the period for $\epsilon=0.2$ (solid line) and $\epsilon=0.75$ (dashed line).

$$0\leqslant|x_n|=\frac{1}{2}(1+\sigma_n\sigma_{n+1}) + \frac{1-2\epsilon}{2(1-\epsilon)}(-\sigma_n\sigma_{n+1})(1-|x_{n+1}|). \tag{19}$$

Except for the fixed points the product $\sigma_n\sigma_{n+1}$ takes the value -1 for at least one n and the condition (19) is violated if $\epsilon\gtrsim 1/2$. Otherwise, if $\epsilon<1/2$ then Eq. (19) yields a contraction on $[0,1]$ so that all symbol sequences are allowed. Hence if p is a prime factor of $T+1$ then no prime orbit of period p appears if $\epsilon\gtrsim 1/2$ but all prime orbits of period p appear if $\epsilon<1/2$. Therefore, pruning rules depend sensitively on the fine tuning of the coefficients of the delay term. Fortunately these features do not corrupt the upper bound for the topological entropy, but an accurate estimation of the entropy from counting periodic orbits seems to be difficult. In the Fig. 2 these pruning rules can be seen in the case of $T=11$.

IV. HÉNON MAP

To gain a little bit more insight into the geometry of delayed maps we investigate a modification of the well-known Hénon map with delayed feedback

$$x_{n+1}=a-x_n^2+bx_{n-T}. \tag{20}$$

It corresponds to the usual Hénon map if $T=1$. Our goal is again to study the properties of periodic orbits as the delay changes. In particular, we will determine their number for estimating the topological entropy and study their stability.

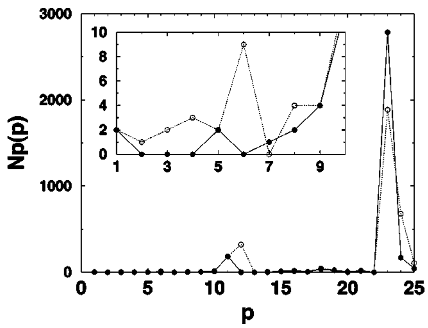

FIG. 2. Number of prime orbits as a function of the period $Np(p)$ for the map (16) for $T=11$, $\epsilon=0.45$, and $\epsilon=0.5$ (empty and filled circles, respectively).

We have used the method proposed in [17] to compute the periodic orbits of the map (20), which was shown to be valid for the normal Hénon map for low values of the parameter b in [18]. The method is originally proposed for the two-dimensional map, but is easily extended to be applied in this system. We have evidence that most of the orbits can be recovered by the modified method, but we cannot guarantee that all orbits are really detected. Moreover, we have no proof that a binary partition (a partition consisting of two elements [16,18]) exists for the high dimensional case, which is one of the requirements for this method to work. Despite of these potential problems, we were able to obtain estimates of the topological entropy from Eq. (7) that we show in Table I for different delay values. Comparing the estimated topological entropies with the metric entropies calculated from the Lyapunov exponents we observe that the values agree within the error bars (except for $T=6$, but in this case the inequality $h_{top} \geqslant h_{ks}$ is also observed).

Although these results are limited to relatively low delay values, they show an important property: the topological entropy agrees with the value of the corresponding metric entropy and moreover, its value does not grow as the delay grows but seems to be bounded. In that respect the model seems to share the properties of the Bernoulli system.

In order to understand why the error bars in Table I increase with the delay, lets us inspect the convergence properties of the topological entropy. Estimates for the topological entropy were obtained using the data sets containing the periods and the respective number of periodic points. Truncating the set at a given p and fitting the data to Eq. (9), we obtain the estimates for the entropy—$h(T,p)$ shown in Fig. 3.

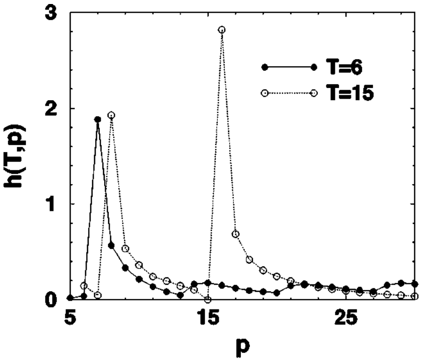

FIG. 3. Convergence of the topological entropy with finite values of period for $T=6$ (solid line) and $T=15$ (dashed line) for the Hénon map with $a=1.0$, $b=0.3$.

For the delay value $T=6$ in Fig. 3 one can see that $h(T,p)$ has a large peak at $p=T+1$ followed by oscillations around an average value. The same pattern was observed for other low delay values not shown in Fig. 3 and seems to be a general feature for this model. Therefore, we can expect satisfactory convergence of the topological entropy only for $p>T+1$. The results in Table I were obtained by the average of all values $h(T,p)$ such that $p>T+1$.

The observed pattern is quite similar to the case of the Bernoulli maps. It has its origin in the number of periodic points of period $p=T+1$. In fact, $N(p=T+1,T)$ equals 2^p as can be evaluated from Eq. (20). For orbits with period $p=T+1$ Eq. (20) reduces to

$$(1-b)\overline{x}_{n+1}=a-\overline{x}_n^2. \tag{21}$$

After linear rescaling, Eq. (21) can be cast into the form of a single logistic map with parameter $a/(1-b)^2$. As long as $a/(1-b)^2>2$ this map has a full set of periodic points giving rise to the just-mentioned phenomenon.

For $p=n(T+1)$ there are also local maxima in the number of periodic points but their values are much smaller than $2^{n(T+1)}$ and we could not find a simple general rule for them. In Fig. 4, these features are illustrated for delays 6 and 15.

TABLE I. Topological and metric entropies of the model (20) with parameters $a=1.0$ and $b=0.3$. Metric entropies have been estimated from the Lyapunov spectrum using Pesins identity.

T	h_{top}	h_{ks}
3	0.21 ± 0.05	0.194
5	0.14 ± 0.02	0.134
6	0.13 ± 0.03	0.085
8	0.13 ± 0.04	0.120
10	0.11 ± 0.04	0.120
12	0.13 ± 0.04	0.114
15	0.09 ± 0.04	0.116

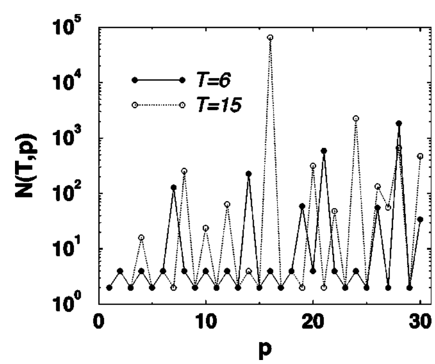

FIG. 4. Number of p-periodic points for $T=6$ (filled circles) and $T=15$ (empty circles) of Eq. (20) with $a=1.0$, $b=0.3$.

TABLE II. Total number of prime orbits of period $p[Np(p)]$ and number of orbits with two-dimensional unstable manifold (N_2) for the Hénon map with $a=1.0$, $b=0.3$, and $T=10$. (Prime orbits are those that do not consist on repetitions of cycles of shorter period.)

p	$Np(p)$	N_2
1	2	0
2	1	1
11	186	0
15	6	0
22	128	6

So far, our results on the topological properties are quite in line with the investigation of the Bernoulli maps. Let us now have a closer look on the stability properties of certain periodic orbits of the map (20). This is neither necessary nor interesting in the case of Bernoulli maps since they have a constant Jacobian and all the orbits have the same stability properties.

Due to the property that any p-periodic orbit existent for delay T, will be also a p-periodic orbit for delay $T+np$, we can study how the stabilities of these orbits change when the delay increases according to this rule. The stabilities of the orbits are fully described by their $T+np+1$ Lyapunov exponents, with $n=0,1,2,\ldots$. The main result is that we can divide the orbits into two categories according to the behavior of the dimension of their unstable manifolds (number of positive Lyapunov exponents) under variation of n.

On the one hand, there are orbits for which the dimension of the unstable manifold does not increase with the delay. As an example let us consider the periodic orbits for the map (20), with $a=1.0$, $b=0.3$, and $T=10$. In Table II we show the periodic orbits detected by the Biham-Wenzel method up to period $p=29$. We have observed that in this case, the orbits, whose unstable manifolds have dimension equal to one, are the vast majority of the computed orbits. For many of these orbits the dimension of the unstable manifold does not change if the delay is increased from T to $T+p$, the corresponding positive Lyapunov exponent stays isolated, and the remaining part of their spectrum is negative. Adopting the same notation as in [11] we may call such an unstable exponent an anomalous one.

On the other hand there exist orbits for which the dimension of the unstable manifold increases with the delay. These orbits show up at $a=1.0$, $b=0.3$, and $T=10$ with a two-dimensional unstable manifold. Although their number is very small compared to the orbits with one-dimensional unstable manifold, the second kind of orbits constitute in general the less unstable. In particular, we find an increasing unstable dimension as the delay increases. For instance, Fig. 5 displays this increase for the period 2 orbit of Table II that exists for all-even values of the delay.

All these observations are on a qualitative level in accordance with the analysis of the Bernoulli map [11]. Whenever the quasicontinuous spectrum of exponents contributes to the unstable exponents, the dimension of the unstable manifold increases linearly with delay, whereas the dimension stays finite if only the anomalous exponent determines the unstable manifold. Since the Hénon map has nonconstant Jacobian, both types of orbits may appear for the same parameter setting simultaneously.

Let us mention finally how the characteristics of the full chaotic attractor of the map at our parameter settings changes

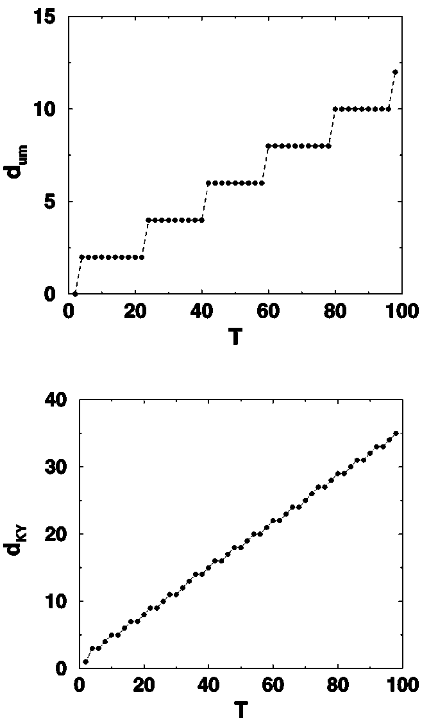

FIG. 5. Dimension of the unstable manifold d_{um} (above) and Kaplan-Yorke dimension d_{ky} (below) for the period-2 orbit of Eq. (20) with $a=1.0$, $b=0.3$, existent for even values of T as a function of the delay.

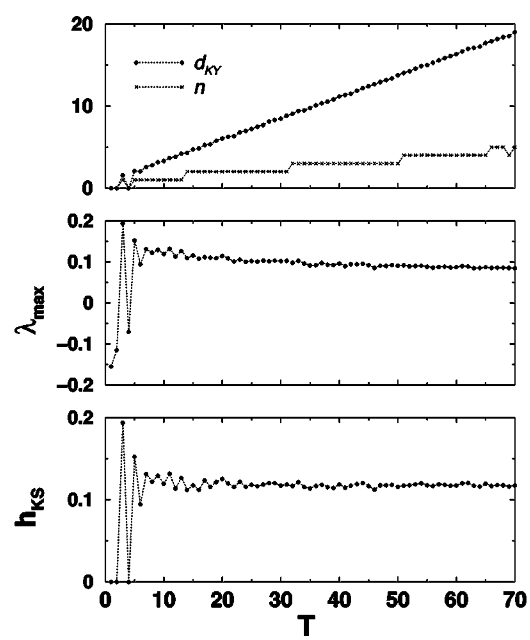

FIG. 6. Dependence of the Kaplan-Yorke dimension d_{ky} and the number of unstable Lyapunov exponents on the delay T for the map (20) with $a=1.0$, $b=0.3$.

with the delay value (cf. Fig. 6). The number of positive Lyapunov exponents grows as the delay increases although the increase is weaker than for the Kaplan-Yorke dimension. This is of course not astonishing since the weakly stable directions also contribute to the Kaplan-Yorke dimension. It would be tempting to find out whether the main contribution to the dimension of the attractor comes from orbits with increasing unstable dimension or from the weakly stable quasicontinuous part of the spectrum.

V. CONCLUSIONS

In summary, we derived some properties of periodic orbits of delayed maps that allowed us to propose two arguments for the boundedness of the topological entropy in the high delay limit. The first argument is more general and could apply also to continuous time delayed systems, as the relation (6) is also valid in this case. The second argument is restricted to piecewise linear maps, but it gives a more insightful understanding: if the generating partition of a piecewise linear map is binary at low delays it will be also binary at higher delays. In other words, the number of elements of the generating partition, in this case, is not affected by changing the delay value. This argument gives naturally an upper bound for the topological entropy, namely the logarithm of the number of elements of the partition. For a binary partition, the upper bound is $h_{top} \leqslant \ln 2$. We should remark, however, that the grammar (the pruning rules governing the existence or absence of a given sequence) changes as the delay is varied. By analyzing numerically the delayed Hénon map, we are tempted to believe that the existence of a finite partition is not altered by changing the delay also in the case of nonlinear maps.

There are several natural extensions of this work possible that go beyond a pure topological characterization of systems with delay. One may check in terms of cycle expansions to what extent periodic orbits with different dimensions of unstable manifolds contribute to the dynamics of chaotic attractors. Since certain types of periodic orbits exist regardless of the specific delay value it is tempting to check whether metric properties like Lyapunov exponents, Kolmogorov Sinai entropies, and dimensions are mainly influenced by these orbits in the high delay limit. This is a natural assumption as the orbits reappear at different delay values, but yet we do not have no proof for such statements.

ACKNOWLEDGMENTS

We are grateful to R. Hegger for the valuable suggestions. E.F.M. acknowledges the financial support from DAAD.

[1] M. C. Mackey and L. Glass, Science **197**, 287 (1977).
[2] J. Foss, A. Longtin, B. Mensour, and J. Milton, Phys. Rev. Lett. **76**, 708 (1996).
[3] N. MacDonald, *Biological Delay Systems: Linear Stability Theory* (Cambridge University Press, Cambridge, England, 1989).
[4] F. T. Arecchi, W. Gadomski, and R. Meucci, Phys. Rev. A **34**, 1617 (1993).
[5] P. K. Asea and P. J. Zak, J. Econ. Dyn. Control **23**, 1155 (1999); M. C. Mackey, J. Econ. Theory **48**, 497 (1989).
[6] R. Hegger, M. J. Bunner, H. Kantz, and A. Giaquinta, Phys. Rev. Lett. **81**, 558 (1998); R. Hegger, Phys. Rev. E **60**, 1563 (1999).
[7] J. D. Farmer, Physica D **4**, 366 (1982).
[8] P. Grassberger and I. Procaccia, Physica D **9**, 189 (1983).
[9] K. Ikeda and K. Matsumoto, J. Stat. Phys. **44**, 955 (1986).
[10] M. Le Berre, E. Ressayre, A. Tallet, H. M. Gibbs, D. L. Kaplan, and M. H. Rose, Phys. Rev. A **35**, 4020 (1987).
[11] S. Lepri, A. Politi, and F. T. Arecchi, Physica D **70**, 234 (1993).
[12] D. Auerbach, P. Cvitanović, J. P. Eckmann, G. Gunaratne, and I. Procaccia, Phys. Rev. Lett. **58**, 2387 (1987). P. Cvitanović, *ibid.* **61**, 2729 (1988); R. Artuso, E. Aurell, and P. Cvitanović, Nonlinearity **3**, 325 (1991).
[13] Y. C. Lai, Y. Nagai, and C. Grebogi, Phys. Rev. Lett. **79**, 649 (1997).
[14] The equations that determine a periodic orbit of Eq. (1) with period θ can be written decomposing the orbit $\gamma(t)$ in a Fourier series. The orbit must satisfy $\dot{\gamma}(t) = F(\gamma(t), \gamma(t-\tau))$. But from the Fourier series we have $\gamma(t-\tau) = \gamma(t-\tau-n\theta)$ and the equations of the orbit are invariant under the transformation $\tau \rightarrow \tau + n\theta$.
[15] P. Walters, *An Introduction to Ergodic Theory* (Springer, New York, 1982).
[16] P. Collet and J. P. Eckmann, *Iterated Maps of the Interval as Dynamical Systems* (Birkhäuser, Stuttgart, 1996).
[17] O. Biham and W. Wenzel, Phys. Rev. Lett. **63**, 819 (1989).
[18] P. Grassberger, H. Kantz, and U. Moenig, J. Phys. A **22**, 5217 (1989).